The classical mechanics from the quantum equation

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Abstract. The quantum hydrodynamic analogy (QHA) is derived as the deterministic limit of its stochastic version. On large scale, the quantum stochastic hydrodynamic analogy (QSHA) shows dynamics that may acquire the classic behavior. The QSHA shows that in presence of spatially distributed noise the quantum behavior is maintained on a distance shorter than the correlation length (λ_c) of fluctuations of the modulus of the wave function. The quantum mechanics is achieved in the deterministic limit when λ_c tends to infinity with respect to the scale of the problem. Moreover, when, the physical length of the problem is of order or larger than λ_c, the model shows that the quantum potential may have a finite range of efficacy maintaining its non-local effect on a finite distance λ_L ("quantum non-locality length"). The paper also unveils that the SQHA has the corresponding stochastic Schrödinger equation as happens for the respective deterministic limits. In the case when the classical limit is approached, the model shows that the dynamics can be described by a non-linear stochastic Schrödinger equation at the glance with the current theoretical outputs. In particular, the work shows that the semi-empirical Gross-Pitaevskii equation describing the ^{4}He dimer gets a theoretical support by the present approach.

1. Introduction

The emergence of classical behavior from a quantum system is a problem of interest in many branches of physics. The incompatibility between the quantum and classical mechanics comes mainly from the non local character of the quantum mechanics. From the empirical point of view, may authors have shown that fluctuations may destroy quantum coherence and elicit the emergence of the classical behavior. By using the alternative approach of the quantum hydrodynamic analogy (QHA) [1] in this paper we investigate how the fluctuations influence the quantum non locality and possibly lead to the large-scale classical evolution.

The motivation of using the quite unknown QHA relies in the fact that it owns a classical-like structure that makes it suitable for the achievement of a comprehensive understanding of quantum and classical phenomena. The suitability of the classical-like theories in explaining open quantum phenomena is a matter of fact and is confirmed by their success in the description of the dispersive effects in semiconductors, multiple tunneling, mesocopic and quantum Brownian oscillators, critical phenomena, and the theoretical regularization procedure of quantum field.

Compared to others classical-like approaches (e.g., the stochastic quantization procedure of Nelson [2] and the mechanics given by Bohm [3]) the QHA has the precious property to be exactly equivalent to the Schrödinger equation and it is free from problems such as the unclear relation between the statistical and the quantum fluctuations as in the Nelson theory or the undefined variables of the Bohmian mechanics. Concerning the last point, as clearly shown by Tsekov [4], it must be noted that the QHA has not to be confused with the Bohmian mechanics.

Among the objectives that could benefit from the present work there are: The clarification of the hierarchy between the classical and quantum mechanics; The achievement of a consistent theory of quantum gravity; The quantum treatment of chaotic dynamical systems and irreversibility.

2. The QSHA equation of motion

When the noise is a stochastic function of the space, in the quantum hydrodynamic analogy the motion equation is described by the stochastic partial differential equation (SPDE) that reads [5]

$$\partial_t \mathrm{n}_{(q,t)} = -\nabla_q \bullet (\mathrm{n}_{(q,t)} \dot{q}) + \eta_{(q,t,\Theta)} \tag{1}$$

$$\dot{p} = -\nabla_q (V_{(q)} + V_{qu(\mathrm{n})}), \tag{2}$$

$$\dot{q} = \frac{\nabla_q S}{m} = \frac{p}{m}, \tag{3}$$

$$S = \int_{t_0}^{t} dt \left(\frac{p \bullet p}{2m} - V_{(q)} - V_{qu(\mathrm{n})} \right) \tag{4}$$

where we generally pose

$$< \eta_{(q_\alpha)}, \eta_{(q_\beta + \lambda)} > = < \eta_{(q_\alpha)}, \eta_{(q_\alpha)} > G(\lambda) \delta_{\alpha\beta} \tag{5}$$

Close to the quantum deterministic limit the PDF field is developed in a series of approximation $\mathrm{n} \cong \mathrm{n}_0 + \Delta \mathrm{n}_1 + \Delta \mathrm{n}_2 + \ldots + \Delta \mathrm{n}_\mathrm{n}$. The condition that the fluctuations of the quantum potential $V_{qu(\mathrm{n})}$ do not diverge, as Θ goes to zero (so that the energy of the fluctuating state does not diverge) is implemented by operating on the system of the discrete version of the SPDE (1) whose variable reads

$$Y_{\boldsymbol{i}} = \iiint_{\Delta_i} d^{3n} q \;\; n_{(q,t)}, \tag{6}$$

where the space hyper-cell $\Delta_i = [(\mathbf{q}_{(i-1)}, (\mathbf{q}_{(i)})]$ (with $\mathbf{q}_{(i)} - \mathbf{q}_{(i-1)} = \lambda$) is taken around the discrete point $q_{(\mathbf{i})}$. The quantum potential fluctuations are derived as a function of the fluctuations of the PDF field at the smallest order $\mathrm{n}_0 + \Delta \mathrm{n}_1$. The results show that in order to have $\lim_{\lambda \to 0} < V^{qu}, V^{qu} >$ finite, the following conditions must be fulfilled

$$\lim_{\lambda \to 0} \sum_\alpha \lambda^{-2} [1 - G_\alpha(\lambda)] < \infty \tag{7}$$

$$\lim_{\lambda \to 0} \sum_\alpha \lambda^{-4} [1 - G_{(\lambda)}]^2 < \infty \tag{8}$$

$$\lim_{\lambda \to 0} \sum_\alpha \lambda^{-4} [3 + G_{(2\lambda)} - 4 G_{(\lambda)}] < \infty \tag{9}$$

Developing G (λ) for small Θ in series expansion as a function of λ/λ_c, where λ_c is defined further on, we obtain

$$\lim_{\lambda \to 0} G_{(\lambda)} \cong a_0 + a_1 \frac{\lambda}{\lambda_c} + a_2 \left(\frac{\lambda}{\lambda_c}\right)^2 + a_3 \left(\frac{\lambda}{\lambda_c}\right)^3 + a_4 \left(\frac{\lambda}{\lambda_c}\right)^4 + \sum_{j=5}^{\infty} a_j \left(\frac{\lambda}{\lambda_c}\right)^j, \tag{10}$$

from where it follows that (7-9) are verified if $a_0 = 1$, $a_1 = 0$, and $a_3 = 0$, while no condition applies to the coefficients a_2 and a_n with $n \geq 4$ that are unable to produce the divergence of (7-9) and remain undefined. Therefore, $G(\lambda)$ reads

$$\lim_{\lambda \to 0} G_{(\lambda)} \cong 1 + a_2 (\frac{\lambda}{\lambda_c})^2 + a_4 (\frac{\lambda}{\lambda_c})^4 + \sum_{j=5}^{\infty} a_j (\frac{\lambda}{\lambda_c})^j . \quad (11)$$

where without a leaking of generality we can put $a_2 = \pm 1$ by a re-definition of the spatial cell side λ such as $\lambda' = {a_2}^{1/2} \lambda$.

In order to obtain a model holding also for a large-scale approach, hence, we investigate in detail the model with $a_2 = -1$ ($a_2 = 1$ does not warrant the ergodicity) with the shape of the correlation function that reads

$$G_{(\lambda)} = exp[-(\frac{\lambda}{\lambda_c})^2] . \quad (12)$$

Thence, from (5) we obtain

$$< \eta_{(q_\alpha, t)}, \eta_{(q_\beta + \lambda, t + \tau)} > = \underline{\mu} \frac{8m(k\Theta)^2}{\pi^3 \hbar^2} exp[-(\frac{\lambda}{\lambda_c})^2] \delta(\tau) \delta_{\alpha\beta} \quad (13)$$

where [5]

$$\lambda_c = (\frac{\pi}{2})^{3/2} \frac{\hbar}{(2mk\Theta)^{1/2}} \quad (14)$$

2.1. Quantum non-locality length λ_L

In addition to the noise correlation function (12), to obtain the macro-scale form of equations (15-23) we need to investigate the large-scale limit of the quantum force $\dot{p}_{qu} = -\nabla_q V_{qu}$ in (2).

As shown in reference [5] the relevance of the quantum potential force at large distance can be evaluated by the convergence of the integral

$$\int_0^{\infty} | q^{-1} \nabla_q V_{qu} | dq < \infty \quad (15)$$

So that the quantum potential range of interaction can be obtained as the mean weighted distance

$$\lambda_L = 2 \frac{\int_0^{\infty} | q^{-1} \frac{dV_{qu}}{dq} | dq}{\lambda_c^{-1} | \frac{dV_{qu}}{dq} |_{(q=\lambda_c)}} . \quad (16)$$

Naming $\Delta\Omega_L$ the physical length of the problem, depending by the two lengths λ_c and λ_L the limiting dynamics follow:

2.2. Schrödinger equation from SQHA

For $\Theta = 0$ equation (1-3) with the identities

$$\dot{q} = \frac{\nabla_q S}{m} \tag{17}$$

where

$$S = \int_{t_0}^{t} dt\left(\frac{p \bullet p}{2m} - V_{(q)} - V_{qu}\right) \tag{18}$$

and

$$\mathrm{n}_{(q,t)} = A^2_{(q,t)} \tag{19}$$

can be derived [35] by the system of two coupled differential equations that read

$$\partial_t \mathrm{S}_{(q,t)} = -V_{(q)} + \frac{\hbar^2}{2m}\frac{\nabla_q{}^2 \mathrm{A}_{(q,t)}}{\mathrm{A}_{(q,t)}} - \frac{1}{2m}(\nabla_q \mathrm{S}_{(q,t)})^2 \tag{20}$$

$$\partial_t \mathrm{A}_{(q,t)} = -\frac{1}{m}\nabla_q \mathrm{A}_{(q,t)} \bullet \nabla_q \mathrm{S}_{(q,t)} - \frac{1}{2m} A \nabla_q{}^2 \mathrm{S}_{(q,t)} \tag{21}$$

that for the complex variable

$$\psi_{(q,t)} = \mathrm{A}_{(q,t)} \exp\left[\frac{\mathrm{i}}{\hbar}\mathrm{S}_{(q,t)}\right] \tag{22}$$

is equivalent to set to zero the real and imaginary part of the Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_q{}^2\psi + V_{(q)}\psi \tag{23}$$

For $\Theta \neq 0$ the stochastic equations (1-3) can be derived by the following system of differential equations

$$\partial_t \mathrm{S}_{(q,t)} = -V_{(q)} + \frac{\hbar^2}{2m}\frac{\nabla_q{}^2 \mathrm{A}_{(q,t)}}{\mathrm{A}_{(q,t)}} - \frac{1}{2m}(\nabla_q \mathrm{S}_{(q,t)})^2 \tag{24}$$

$$\partial_t \mathrm{A}_{(q,t)} = -\frac{1}{m}\nabla_q \mathrm{A}_{(q,t)} \bullet \nabla_q \mathrm{S}_{(q,t)} - \frac{1}{2m} A \nabla_q{}^2 \mathrm{S}_{(q,t)} + A^{-1}\eta_{(q_\alpha,t,\Theta)} \tag{25}$$

that for the complex variable (22) are equivalent to the stochastic Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_q{}^2\psi + V_{(q)}\psi + i\frac{\psi}{|\psi|^2}\eta_{(q_\alpha,t,\Theta)}. \tag{26}$$

2.3. Limiting dynamics

1) *Non-local deterministic dynamics* (i.e., the standard quantum mechanics) with $\Delta\Omega_L << \lambda_c \cup \lambda_L$ (i.e., $\Theta \rightarrow 0$):

$$\partial_t n_{0(q,t)} = -\nabla_q \bullet (n_{0(q,t)} \dot{q}) \tag{27}$$

2) *Non-local stochastic dynamics*, with $\lambda_c << \Delta\Omega_L << \lambda_L$

$$\partial_t n_{(q,t)} = -\nabla_q \bullet (n_{(q,t)} \nabla_q \dot{q}) + \eta_{(q_\alpha ,t,\Theta)} \tag{28}$$

$$< \eta_{(q_\alpha ,t)}, \eta_{(q_\alpha + \lambda ,t+\tau)} > = \underline{\mu}\, \delta_{\alpha\beta} \frac{2k\Theta}{\lambda_c} \delta(\lambda) \delta(\tau) \tag{29}$$

In this case the stochastic Schrödinger equation (26) reads

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla_q{}^2 \psi + V_{(q)} \psi + i \frac{\psi}{|\psi|^2} \eta_{(q_\alpha ,t,\Theta)} \tag{30}$$

$$< \eta_{(q_\alpha ,t)}, \eta_{(q_\alpha + \lambda ,t+\tau)} > = \underline{\mu}\, \delta_{\alpha\beta} \frac{2k\Theta}{\lambda_c} \delta(\lambda) \delta(\tau) \tag{31}$$

3) *Local stochastic dynamics*, with $\lambda_c \cup \lambda_L << \Delta\Omega q << \Delta\Omega_L$.

Given the condition $\lambda_L << \Delta\Omega q << \Delta\Omega_L$ so that it holds

$$\lim_{q \to \infty} -\nabla_q V_{qu(n_0)} = 0 \tag{32}$$

the SPDE of motion acquires the form

$$\partial_t n_{(q,t)} = -\nabla_q \bullet (n_{(q,t)} \dot{q}) + \eta_{(q_\alpha ,t,\Theta)} \tag{33}$$

$$< \eta_{(q_\alpha ,t)}, \eta_{(q_\alpha + \lambda ,t+\tau)} > = \underline{\mu}\, \delta_{\alpha\beta} \frac{2k\Theta}{\lambda_c} \delta(\lambda) \delta(\tau) \tag{34}$$

$$\begin{aligned} \dot{q} &= \frac{p}{m} = \nabla_q \lim_{\Delta\Omega / \lambda_L \to \infty} \frac{\nabla_q S}{m} = \nabla_q \{ \lim_{\Delta\Omega / \lambda_L \to 0} \frac{1}{m} \int_{t_0}^{t} dt(\frac{p \bullet p}{2m} - V_{(q)} - V_{qu(n)} - I^*) \} \\ &= \frac{1}{m} \nabla_q \{ \int_{t_0}^{t} dt \, (\frac{p \bullet p}{2m} - V_{(q)} - \Delta) \} = \frac{p_{cl}}{m} + \frac{\delta p}{m} \cong \frac{p_{cl}}{m} \end{aligned} \tag{35}$$

where δp is a small fluctuation of momentum and

$$\dot{p}_{cl} = -\nabla_q V_{(q)} . \tag{36}$$

In this case, by using the identities (17-19) we can write

$$\partial_t S = -V_{(q)} - \frac{1}{2m} (\nabla_q S)^2 \tag{37}$$

$$\partial_t A_{(q,t)} = -\frac{1}{m}\nabla_q A_{(q,t)} \bullet \nabla_q S_{(q,t)} - \frac{1}{2m} A \nabla_q{}^2 S_{(q,t)} + A^{-1}\eta_{(q_\alpha,t,\Theta)} \quad (38)$$

Clearly, it is not possible obtain the Schrödinger equation by (37-38) since S given by (35) converges to the classical value S_{cl}

$$S_{cl} = \int_{t_0}^{t} dt \, (\frac{p \bullet p}{2m} - V_{(q)}). \quad (39)$$

Nevertheless, for the wave function (22) the classical stochastic equation of motion (37-38) can be cast in a non-linear Schrödinger equation that reads:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla_q{}^2\psi - \frac{\psi}{|\psi|}\nabla_q{}^2 |\psi|) + V_{(q)}\psi + i\frac{\psi}{|\psi|^2}\eta_{(q_\alpha,t,\Theta)}. \quad (40)$$

The former differential equation describe the evolution of a particle spatial density |ψ| owing a classical action S_{cl}. Actually, the exact equation is given by (26) while the former one (40) is just a limiting one and the formal transformation between them is just intrinsic.

Thence, in order to describe phenomena at the edge between the classical and the quantum behavior, a more manageable semi-empirical equation for passing from (26) to (40) can be useful.

By considering that the when the physical length of the system $\Delta\Omega_L$ is much smaller than the quantum non-locality length λ_L, the system is quantum, while when λ_L is very small compared to $\Delta\Omega_L$ is classic, it is possible to write

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla_q{}^2\psi - \alpha\frac{\psi}{|\psi|}\nabla_q{}^2 |\psi|) + V_{(q)}\psi + i\frac{\psi}{|\psi|^2}\eta_{(q_\alpha,t,\Theta)} \quad (41)$$

where α at first order in a series expansion as a function of the dimensionless parameter $\frac{\lambda_L}{\Delta\Omega_L}$ reads

$$\alpha \cong \frac{\frac{\Delta\Omega_L}{\lambda_L}}{1+\frac{\Delta\Omega_L}{\lambda_L}} \quad (42)$$

and where λ_L is given by (16).

It is interesting to note that Equation (41) for pseudo-Gaussian states that have large-distance behavior such as

$$\lim_{q\to\infty} |\psi| \propto exp[-kq]$$

acquires the large-distance form

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}(\nabla_q{}^2\psi-A\,|\,\psi\,|^m\;\psi\,)+V_{(\,q\,)}\psi+i\frac{\psi}{|\,\psi\,|^2}\eta_{(\,q_\alpha,t,\Theta\,)} \tag{43}$$

with $m = 0$, A= $\alpha \mathrm{k}^2$. For states that have the large-distance hyperbolic behavior

$$\lim_{q\to\infty}\;|\,\psi\,|\propto l^{-1/2}q^{-1} \tag{44}$$

that holds for Lennard-Jones type potentials

$$V_{L-J(\,q\,)}=4V_0\,[\,(\frac{\sigma}{r})^{12}-(\frac{\sigma}{r})^6\,] \tag{45}$$

such as in the ^{4}He dimer [37] equation (41) acquires the stochastic form of the Gross-Pitaevskii equation [38] (i.e., m=2, A= l /2)

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}(\nabla_q{}^2\psi-\alpha\frac{l}{2}\,|\,\psi\,|^2\;\psi\,)+V_{(\,q\,)}\psi+i\frac{\psi}{|\,\psi\,|^2}\eta_{(\,q_\alpha,t,\Theta\,)} \tag{46}$$

3. Discussion

The existence of λ_L finite allows fluctuations, as small as we like, to overcome the "regular" quantum force on large distance so that the quantum non-locality can only be maintained on a finite distance of order of λ_L. Since λ_L finite can steam out from a large number of real non-linear potentials, while the case of an infinite quantum non-locality length (such as in the linear case) actually seems to be an exception, the universe behaves classic on its huge scale. Generally speaking, it must be observed that even thought fluctuations are present, we may have systems characterized by an infinite quantum non-locality length λ_L (e.g., linear systems owing Gaussian states) so that fluctuations are not sufficient to break the quantum mechanics and to lead to the classical one. Under this light, the macro-scale description is not sufficient to obtain the classical behavior if not coupled to a finite quantum non-locality. With this respect, the WKB approximation is an illuminating example being the non-local large-scale description but the classical limit. On the contrary, fluctuations may break quantum non-locality in non-linear systems (λ_L finite) because, in this case, the quantum pseudo-potential decreases with distance and, beyond the non-locality length λ_L, it becomes much smaller than the noise and can be neglected. It must be noted that, only in the stochastic approach the quantum potential can be correctly neglected while it cannot be taken off by the deterministic limit of equation (1-3) because in such a case this operation will change the structure of the equation [5] destroying the quantum stationary states (i.e., eigenstates) and deeply changing the evolution of the system in a sufficiently short interval of time.

4. Conclusions

The investigation of the QSHA shows that the quantum potential in presence of spatial noise is source of fluctuations that modifies the shape of the fluctuations of the PDF field (whose spatial density in the deterministic limit represents the wave function modulus) suppressing them on a distance much shorter than the theory-defined quantum coherence length $\lambda_c=(\frac{\pi}{2})^{3/2}\frac{\hbar}{(\,2mk\Theta\,)^{1/2}}$ so that the quantum mechanics is achieved when λ_c goes to infinity with respect to physical scale of the problem (or in the deterministic limit of null noise amplitude $\Theta = 0$). The correlation function of the PDF field fluctuations (and its characteristic distance

λ_c) close to the deterministic limit of standard quantum mechanics, has been derived by imposing that the system energy in the fluctuating state does not diverge but remains finite. The model highlights that in the stochastic case, beyond the quantum coherence length λ_c, the quantum potential may have a finite range of efficacy maintaining the non-local behavior on a distance of order of the theory-defined "quantum non-locality length" λ_L depending both by fluctuations amplitude and by the inter-particle law of interaction. Generally speaking, it has been shown that fluctuations are not sufficient to break the non-local quantum character. In the case of linear systems it has been shown that $\lambda_L = \infty$ even if λ_c is finite. For non-linear interactions, the noise may produce quantum non-locality breaking when the force of the quantum potential decreases and becomes vanishing at large distance (beyond λ_L finite) becoming negligible with respect to the fluctuations. For $\hbar \neq 0$ the classical stochastic behavior is achieved when λ_c as well as λ_L are negligibly small with respect to the physical length of the problem, while the deterministic classical limit is realized only for the unphysical case of $\hbar = 0$. The QSHA model furnishes the logical compatibility between the quantum and the classical behavior in the frame of a unique approach. The quantum mechanics is deterministic (at glance with satisfying philosophical requirements of the quantum mechanics) while the classical one is achieved when, beyond λ_L, fluctuations disrupt the quantum potential action as well as the quantum eigenstates that it builds up.
Moreover, the SQHA is able to give a theoretical support to the formulation of the semi-empirical non-linear Schrödinger equations (such as the Gross-Pitaevskii one) needed to describe the open quantum mechanics where, actually, the classical effects start to be relevant.

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